

# A fundamental solution to the time-periodic Stokes equations

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The concept of a fundamental solution to the time-periodic Stokes equations in dimension  $n \geq 2$  is introduced. A fundamental solution is then identified and analyzed. Integrability and pointwise estimates are established.

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## 1 Introduction

Classically, fundamental solutions are defined for systems of linear partial differential equations in  $\mathbb{R}^n$ . Specifically, a fundamental solution to the Stokes system ( $n \geq 2$ )

$$\begin{cases} -\Delta v + \nabla p = f & \text{in } \mathbb{R}^n, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^n, \end{cases} \quad (1.1)$$

with unknowns  $v : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $p : \mathbb{R}^n \rightarrow \mathbb{R}$  and data  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , is a tensor-field

$$\Gamma_{\text{Stokes}} := \begin{pmatrix} \Gamma_{11}^s & \cdots & \Gamma_{1n}^s \\ \vdots & \ddots & \vdots \\ \Gamma_{n1}^s & \cdots & \Gamma_{nn}^s \\ \gamma_1^s & \cdots & \gamma_n^s \end{pmatrix} \in \mathcal{S}'(\mathbb{R}^n)^{(n+1) \times n}$$

that satisfies<sup>1</sup>

$$\begin{cases} -\Delta \Gamma_{ij}^s + \partial_i \gamma_j^s = \delta_{ij} \delta_{\mathbb{R}^n}, \\ \partial_i \Gamma_{ij}^s = 0, \end{cases} \quad (1.2)$$

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<sup>1</sup>We make use of the Einstein summation convention and implicitly sum over all repeated indices.

where  $\delta_{ij}$  and  $\delta_{\mathbb{R}^n}$  denotes the Kronecker delta and delta distribution, respectively. For arbitrary  $f \in \mathcal{S}(\mathbb{R}^n)^n$ , a solution  $(v, p)$  to (1.1) is then given by the componentwise convolution

$$\begin{pmatrix} v \\ p \end{pmatrix} := \Gamma_{\text{Stokes}} * f, \quad (1.3)$$

which at the outset is well-defined in the sense of distributions. In the specific case of the Stokes fundamental solution  $\Gamma_{\text{Stokes}}$  above,  $L^q$ -integrability and pointwise decay estimates for  $(v, p)$  can be established from (1.3). We refer to the standard literature such as [3] and [7] for these well-known results.

The aim of this paper is to identify a fundamental solution to the *time-periodic* Stokes system

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathbf{p} = f & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(x, t) = u(x, t + \mathcal{T}) \end{cases} \quad (1.4)$$

with unknowns  $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  and  $\mathbf{p} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  corresponding to time-periodic data  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  with the same period, that is,  $f(x, t) = f(x, t + \mathcal{T})$ . Here  $\mathcal{T} \in \mathbb{R}$  denotes the (fixed) time-period. Moreover,  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}$  denotes the spatial and time variable, respectively. The main objective is to establish a framework which enables us to define and identify a fundamental solution  $\Gamma_{\text{TPStokes}}$  to (1.4) with the property that a solution  $(u, \mathbf{p})$  is given by a convolution

$$\begin{pmatrix} u \\ \mathbf{p} \end{pmatrix} := \Gamma_{\text{TPStokes}} * f. \quad (1.5)$$

Having obtained this goal, we shall then examine to which extent regularity such as  $L^q$ -integrability and pointwise estimates of the solution can be derived from (1.5).

Since time-periodic data  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ ,  $(x, t) \rightarrow f(x, t)$  are non-decaying in  $t$ , a framework based on classical convolution in  $\mathbb{R}^n \times \mathbb{R}$  cannot be applied. Instead, we reformulate (1.4) as a system of partial differential equations on the locally compact abelian group  $G := \mathbb{R}^n \times \mathbb{R}/\mathcal{T}\mathbb{Z}$ . More specifically, we exploit that  $\mathcal{T}$ -time-periodic functions can naturally be identified with mappings on the torus group  $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$  in the time variable  $t$ . In the setting of the Schwartz-Bruhat space  $\mathcal{S}(G)$  and corresponding space of tempered distributions  $\mathcal{S}'(G)$ , we can then define a fundamental solution  $\Gamma_{\text{TPStokes}}$  to (1.4) as a tensor-field

$$\Gamma_{\text{TPStokes}} := \begin{pmatrix} \Gamma_{11}^{\text{TPS}} & \cdots & \Gamma_{1n}^{\text{TPS}} \\ \vdots & \ddots & \vdots \\ \Gamma_{n1}^{\text{TPS}} & \cdots & \Gamma_{nn}^{\text{TPS}} \\ \gamma_1^{\text{TPS}} & \cdots & \gamma_n^{\text{TPS}} \end{pmatrix} \in \mathcal{S}'(G)^{(n+1) \times n} \quad (1.6)$$

that satisfies

$$\begin{cases} \partial_t \Gamma_{ij}^{\text{TPS}} - \Delta \Gamma_{ij}^{\text{TPS}} + \partial_i \gamma_j^{\text{TPS}} = \delta_{ij} \delta_G, \\ \partial_i \Gamma_{ij}^{\text{TPS}} = 0 \end{cases} \quad (1.7)$$

in the sense of  $\mathcal{S}'(G)$ -distributions. A solution to the time-periodic Stokes system (1.4) is then given by (1.5), provided the convolution is taken over the group  $G$ .

The aim in the following is to identify a tensor-field  $\Gamma_{\text{TPStokes}} \in \mathcal{S}'(G)^{(n+1) \times n}$  satisfying (1.7). We shall describe  $\Gamma_{\text{TPStokes}}$  as a sum of the steady-state Stokes fundamental solution  $\Gamma_{\text{Stokes}}$  and a remainder part satisfying remarkably good integrability and pointwise decay estimates. It is well-known that the components of the velocity part  $\Gamma^S \in \mathcal{S}'(\mathbb{R}^n)^{n \times n}$  and pressure part  $\gamma^S \in \mathcal{S}'(\mathbb{R}^n)^n$  of  $\Gamma_{\text{Stokes}}$  are functions

$$\Gamma_{ij}^S(x) := \begin{cases} \frac{1}{2\omega_n} \left( \delta_{ij} \log(|x|^{-1}) + \frac{x_i x_j}{|x|^2} \right) & \text{if } n = 2, \\ \frac{1}{2\omega_n} \left( \delta_{ij} \frac{1}{n-2} |x|^{2-n} + \frac{x_i x_j}{|x|^n} \right) & \text{if } n \geq 3, \end{cases}$$

$$\gamma_i^S(x) := \frac{1}{\omega_n} \frac{x_i}{|x|^n},$$

respectively; see for example [3, IV.2]. Here,  $\omega_n$  denotes the surface area of the  $(n-1)$ -dimensional unit sphere in  $\mathbb{R}^n$ . Our main theorem reads:

**Theorem 1.1.** *Let  $n \geq 2$ . There is a fundamental solution  $\Gamma_{\text{TPStokes}} \in \mathcal{S}'(G)^{(n+1) \times n}$  to the time-periodic Stokes equations (1.4) on the form (1.6) satisfying (1.7) and*

$$\Gamma^{\text{TPS}} = \Gamma^S \otimes 1_{\mathbb{T}} + \Gamma^{\perp}, \quad (1.8)$$

$$\gamma^{\text{TPS}} = \gamma^S \otimes \delta_{\mathbb{T}} \quad (1.9)$$

with  $\Gamma^{\perp} \in \mathcal{S}'(G)^{n \times n}$  satisfying

$$\forall q \in \left(1, \frac{n}{n-1}\right) : \quad \Gamma^{\perp} \in L^q(G)^{n \times n}, \quad (1.10)$$

$$\forall r \in [1, \infty) \quad \forall \varepsilon > 0 \quad \exists C > 0 \quad \forall |x| \geq \varepsilon : \quad \|\Gamma^{\perp}(x, \cdot)\|_{L^r(\mathbb{T})} \leq \frac{C}{|x|^n}, \quad (1.11)$$

$$\forall q \in (1, \infty) \quad \exists C > 0 \quad \forall f \in \mathcal{S}(G)^n : \quad \|\Gamma^{\perp} * f\|_{W^{2,1,q}(G)} \leq C \|f\|_{L^q(G)}, \quad (1.12)$$

where  $\mathbb{T}$  denotes the torus group  $\mathbb{T} := \mathbb{R}/\mathbb{T}\mathbb{Z}$ ,  $1_{\mathbb{T}} \in \mathcal{S}'(\mathbb{T})$  the constant 1,  $\delta_{\mathbb{T}} \in \mathcal{S}'(\mathbb{T})$  the Dirac delta distribution on  $\mathbb{T}$ ,  $*$  the convolution on  $G$ , and  $W^{2,1,q}(G)$  the Sobolev space of order 2 in  $x$  and order 1 in  $t$ .

*Remark 1.2.* We shall briefly demonstrate how the fundamental solution (1.8)–(1.9) can be applied in a more classical setting of the time-periodic Stokes equations to obtain a representation formula, integrability properties and decay estimates of a solution. The time-periodic Stokes equations are typically studied in a function analytical framework based on the function space

$$C_{0,\text{per}}^{\infty}(\mathbb{R}^n \times \mathbb{R}) := \{f \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}) \mid f(x, t + \mathcal{T}) = f(x, t) \wedge f \in C_0^{\infty}(\mathbb{R}^n \times [0, \mathcal{T}])\},$$

upon which  $\|f\|_q := \|f\|_{L^q(\mathbb{R}^n \times [0, \tau])}$  is a norm. Time-periodic Lebesgue and Sobolev spaces are defined as

$$L_{\text{per}}^q(\mathbb{R}^n \times \mathbb{R}) := \overline{C_{0,\text{per}}^\infty(\mathbb{R}^n \times \mathbb{R})}^{\|\cdot\|_q},$$

$$W_{\text{per}}^{2,1,q}(\mathbb{R}^n \times \mathbb{R}) := \overline{C_{0,\text{per}}^\infty(\mathbb{R}^n \times \mathbb{R})}^{\|\cdot\|_{2,1,q}}, \quad \|f\|_{2,1,q} := \left( \sum_{|\alpha| \leq 2} \|\partial_x^\alpha f\|_q^q + \sum_{|\beta| \leq 1} \|\partial_t^\beta f\|_q^q \right)^{\frac{1}{q}}.$$

It is easy to see that  $L_{\text{per}}^q(\mathbb{R}^n \times \mathbb{R})$  and  $W_{\text{per}}^{2,1,q}(\mathbb{R}^n \times \mathbb{R})$  are isometrically isomorphic to  $L^q(G)$  and  $W^{2,1,q}(G)$ , respectively. Regarding  $\Gamma^\perp$  as a tensor-field in  $L_{\text{per}}^q(\mathbb{R}^n \times \mathbb{R})$ , we obtain by Theorem 1.1 for any sufficiently smooth vector-field  $f$ , say  $f \in C_{0,\text{per}}^\infty(\mathbb{R}^n \times \mathbb{R})^n$ , a solution  $(u, \mathbf{p})$  to (1.4) given by  $u := u_1 + u_2$  with

$$u_1 := \left[ \Gamma^{\text{S}} *_{\mathbb{R}^n} \left( \frac{1}{\tau} \int_0^\tau f(\cdot, s) \, ds \right) \right](x, t),$$

$$u_2 := \int_{\mathbb{R}^n} \frac{1}{\tau} \int_0^\tau \Gamma^\perp(x - y, t - s) f(y, s) \, ds dy$$
(1.13)

and  $\mathbf{p}(x, t) := [\gamma^{\text{S}} *_{\mathbb{R}^n} f(\cdot, t)](x)$ . Properties of  $u_1$  and  $\mathbf{p}$  can be derived directly from the Stokes fundamental solution  $(\Gamma^{\text{S}}, \gamma^{\text{S}})$ , which, given the simple structure of  $(\Gamma^{\text{S}}, \gamma^{\text{S}})$ , is elementary and can be found in standard literature such as [3] and [7]. To fully understand the structure of a time-periodic solution, it therefore remains to investigate  $u_2$ . For this purpose, (1.10)–(1.12) of Theorem 1.1 is useful. For example, (1.12) yields integrability  $u_2 \in W_{\text{per}}^{2,1,q}(\mathbb{R}^n \times \mathbb{R})$ , and from (1.11) the pointwise decay estimate  $|u_2(x, t)| \leq C|x|^{-n}$  can be derived for large values of  $x$ .

*Remark 1.3.* Theorem 1.1 implies that  $\Gamma^\perp$  decays faster than  $\Gamma^{\text{S}}$  as  $|x| \rightarrow \infty$ ; both in terms of summability (1.10) and pointwise (1.11). This information provides us with a valuable insight into the asymptotic structure as  $|x| \rightarrow \infty$  of a time-periodic solution to the Stokes equations. More precisely, from the representation formula  $u = u_1 + u_2$  with  $u_1$  and  $u_2$  given by (1.13), and the fact that  $\Gamma^\perp$  decays faster than  $\Gamma^{\text{S}}$  as  $|x| \rightarrow \infty$ , it follows that the leading term in an asymptotic expansion of  $u$  coincides with the leading term in the expansion of  $u_1$ . Since  $u_1$  is a solution to a steady-state Stokes problem, it is well-known how to identify its leading term. In conclusion, Theorem 1.1 tells us that time-periodic solutions to the Stokes equations essentially have the same well-known asymptotic structure as  $|x| \rightarrow \infty$  as steady-state solutions—a nontrivial fact, which is not clear at the outset.

The Stokes system is a linearization of the nonlinear Navier-Stokes system. A fundamental solution to the time-periodic Stokes equations can therefore be used to develop a linear theory for the time-periodic Navier-Stokes problem. The study of the time-periodic Navier-Stokes equations was initiated by SERRIN [6], PRODI [5], and YUDOVICH [8]. Since then, a number of papers have appeared based on the techniques proposed

by these authors. The methods all have in common that the time-periodic problem is investigated in a setting of the corresponding initial-value problem, and time-periodicity of a solution only established a posteriori. With an appropriate time-periodic linear theory, a more direct approach to the time-periodic Navier-Stokes problem can be developed, which may reveal more information on the solutions. The asymptotic structure mentioned in Remark 1.3 is but one example.

## 2 Preliminaries

Points in  $\mathbb{R}^n \times \mathbb{R}$  are denoted by  $(x, t)$  with  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ . We refer to  $x$  as the spatial and to  $t$  as the time variable.

We denote by  $B_R := B_R(0)$  balls in  $\mathbb{R}^n$  centered at 0. Moreover, we let  $B_{R,r} := B_R \setminus \overline{B_r}$  and  $B^R := \mathbb{R}^n \setminus \overline{B_R}$ .

For a sufficiently regular function  $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ , we put  $\partial_i u := \partial_{x_i} u$ . The differential operators  $\Delta$ ,  $\nabla$  and  $\operatorname{div}$  act only in the spatial variables. For example,  $\operatorname{div} u := \sum_{j=1}^n \partial_j u_j$  denotes the divergence of  $u$  with respect to the  $x$  variables.

We let  $G$  denote the group  $G := \mathbb{R}^n \times \mathbb{T}$ , with  $\mathbb{T}$  denoting the torus group  $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ .  $G$  is equipped with the quotient topology and differentiable structure inherited from  $\mathbb{R}^n \times \mathbb{R}$  via the quotient mapping  $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow G$ ,  $\pi(x, t) := (x, [t])$ . Clearly,  $G$  is a locally compact abelian group with Haar measure given by the product of the Lebesgue measure  $dx$  on  $\mathbb{R}^n$  and the (normalized) Haar measure  $dt$  on  $\mathbb{T}$ . We implicitly identify  $\mathbb{T}$  with the interval  $[0, \mathcal{T})$ , whence the (normalized) Haar measure on  $\mathbb{T}$  is determined by

$$\forall f \in C(\mathbb{T}) : \quad \int_{\mathbb{T}} f \, dt := \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} f(t) \, dt.$$

We identify the dual group  $\widehat{G}$  with  $\mathbb{R}^n \times \mathbb{Z}$  and denote points in  $\widehat{G}$  by  $(\xi, k)$ .

We denote by  $\mathcal{S}(G)$  the Schwartz-Bruhat space of generalized Schwartz functions; see [2]. By  $\mathcal{S}'(G)$  we denote the corresponding space of tempered distributions. The Fourier transform on  $G$  and its inverse takes the form

$$\begin{aligned} \mathcal{F}_G : \mathcal{S}(G) &\rightarrow \mathcal{S}(\widehat{G}), \quad \mathcal{F}_G[u](\xi, k) := \int_{\mathbb{R}^n} \int_{\mathbb{T}} u(x, t) \, e^{-ix \cdot \xi - ik \frac{2\pi}{\mathcal{T}} t} \, dt dx, \\ \mathcal{F}_G^{-1} : \mathcal{S}(\widehat{G}) &\rightarrow \mathcal{S}(G), \quad \mathcal{F}_G^{-1}[w](x, t) := \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} w(\xi, k) \, e^{ix \cdot \xi + ik \frac{2\pi}{\mathcal{T}} t} \, d\xi, \end{aligned}$$

respectively, provided the Lebesgue measure  $d\xi$  is normalized appropriately. By duality,  $\mathcal{F}_G$  extends to a homeomorphism  $\mathcal{F}_G : \mathcal{S}'(G) \rightarrow \mathcal{S}'(\widehat{G})$ . Observe that  $\mathcal{F}_G = \mathcal{F}_{\mathbb{R}^n} \circ \mathcal{F}_{\mathbb{T}}$ .

We denote by  $\delta_{\mathbb{R}^n}$ ,  $\delta_{\mathbb{T}}$ ,  $\delta_{\mathbb{Z}}$  the Dirac delta distribution on  $\mathbb{R}^n$ ,  $\mathbb{T}$  and  $\mathbb{Z}$ , respectively. Observe that  $\delta_{\mathbb{Z}}$  is a function with  $\delta_{\mathbb{Z}}(k) = 1$  if  $k = 0$  and  $\delta_{\mathbb{Z}}(k) = 0$  otherwise. Also note that  $\mathcal{F}_{\mathbb{T}}[1_{\mathbb{T}}] = \delta_{\mathbb{Z}}$ .

Given a tensor  $\Gamma \in \mathcal{S}'(G)^{n \times m}$ , we define the convolution of  $\Gamma$  with vector field  $f \in \mathcal{S}(G)^m$  as the vector field  $\Gamma * f \in \mathcal{S}'(G)^n$  with  $[\Gamma * f]_i := \Gamma_{ij} * f_j$ .

The  $L^q(G)$ -spaces with norm  $\|\cdot\|_q$  are defined in the usual way via the Haar measure  $dxdt$  on  $G$ . We further introduce the Sobolev space

$$W^{2,1,q}(G) := \overline{C_0^\infty(G)}^{\|\cdot\|_{2,1,q}}, \quad \|f\|_{2,1,q} := \left( \sum_{|\alpha| \leq 2} \|\partial_x^\alpha f\|_q^q + \sum_{|\beta| \leq 1} \|\partial_t^\beta f\|_q^q \right)^{\frac{1}{q}},$$

where  $C_0^\infty(G)$  denotes the space of smooth functions of compact support on  $G$ .

We emphasize at this point that a framework based on  $G$  is a natural setting for the time-period Stokes equations. It is easy to see that lifting by the restriction  $\pi|_{\mathbb{R}^n \times [0, \mathcal{T})}$  of the quotient mapping provides us with an equivalence between the time-periodic Stokes problem (1.4) and the system

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{in } G, \\ \operatorname{div} u = 0 & \text{in } G. \end{cases}$$

An immediate advantage obtained by writing the time-periodic Stokes problem as system of equations on  $G$  is the ability to then apply the Fourier transform  $\mathcal{F}_G$  and re-write the problem in terms of Fourier symbols. We shall take advantage of this possibility in the proof of the main theorem below.

We use the symbol  $C$  for all constants. In particular,  $C$  may represent different constants in the scope of a proof.

### 3 Proof of main theorem

*Proof of Theorem 1.1.* Put

$$\Gamma^\perp := \mathcal{F}_G^{-1} \left[ \frac{1 - \delta_{\mathbb{Z}}(k)}{|\xi|^2 + i \frac{2\pi}{\mathcal{T}} k} \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \right], \quad (3.1)$$

where  $I \in \mathbb{R}^{n \times n}$  denotes the identity matrix. Since

$$M : \widehat{G} \rightarrow \mathbb{C}, \quad M(\xi, k) := \frac{1 - \delta_{\mathbb{Z}}(k)}{|\xi|^2 + i \frac{2\pi}{\mathcal{T}} k} \quad (3.2)$$

is bounded, that is,  $M \in L^\infty(\widehat{G})$ , the inverse Fourier transform in (3.1) is well-defined as a distribution in  $\mathcal{S}'(G)^{n \times n}$ . Now define  $\Gamma^{\text{TPS}}$  and  $\gamma^{\text{TPS}}$  as in (1.8) and (1.9). It is then easy to verify that  $(\Gamma^{\text{TPS}}, \gamma^{\text{TPS}})$  is a solution to (1.7).

It remains to show (1.10)–(1.12). For this purpose, we introduce for  $k \in \mathbb{Z} \setminus \{0\}$  the function

$$\Gamma_{\text{SSR}}^k : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}, \quad \Gamma_{\text{SSR}}^k(x) := \frac{i}{4} \left( \frac{\sqrt{-i \frac{2\pi}{\mathcal{T}} k}}{2\pi|x|} \right)^{\frac{n-2}{2}} H_{\frac{n}{2}-1}^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}} k} \cdot |x| \right), \quad (3.3)$$

where  $H_\alpha^{(1)}$  denotes the Hankel function of the first kind, and  $\sqrt{z}$  the square root of  $z$  with *positive* imaginary part. As one readily verifies,  $\Gamma_{\text{SSR}}^k$  is a fundamental solution to the Helmholtz equation

$$\left(-\Delta + i\frac{2\pi}{\mathcal{T}}k\right)\Gamma_{\text{SSR}}^k = \delta_{\mathbb{R}^n} \quad \text{in } \mathbb{R}^n. \quad (3.4)$$

Clearly,  $\Gamma_{\text{SSR}}^k \in \mathcal{S}'(\mathbb{R}^n)$ . Moreover, its Fourier transform is given by the function

$$\mathcal{F}_{\mathbb{R}^n}[\Gamma_{\text{SSR}}^k](\xi) = \frac{1}{|\xi|^2 + i\frac{2\pi}{\mathcal{T}}k}. \quad (3.5)$$

From the estimates in Lemma 3.1 below, we see that

$$\int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |\Gamma_{\text{SSR}}^k|^2 \right)^{\frac{q}{2}} dx < \infty$$

for  $q \in (1, \frac{n}{n-1})$ . By Hölder's inequality and Parseval's theorem, we thus deduce

$$\begin{aligned} & \int_{\mathbb{T}} \int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{T}}^{-1}[(1 - \delta_{\mathbb{Z}}(k)) \cdot \Gamma_{\text{SSR}}^k(x)](t)|^q dx dt \\ & \leq C \int_{\mathbb{R}^n} \left( \int_{\mathbb{T}} |\mathcal{F}_{\mathbb{T}}^{-1}[(1 - \delta_{\mathbb{Z}}(k)) \cdot \Gamma_{\text{SSR}}^k]|^2 dt \right)^{\frac{q}{2}} dx \\ & \leq C \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |\Gamma_{\text{SSR}}^k|^2 \right)^{\frac{q}{2}} dx < \infty. \end{aligned}$$

It is well-known that the Riesz transform  $\mathfrak{R}_k(f) := \mathcal{F}_{\mathbb{R}^n}^{-1}[\frac{\xi_k}{|\xi|} \cdot \mathcal{F}_{\mathbb{R}^n}[f]]$  is bounded on  $L^q(\mathbb{R}^n)$  for all  $q \in (1, \infty)$ . Consequently, we obtain  $\mathfrak{R}_i \circ \mathfrak{R}_j(\mathcal{F}_{\mathbb{T}}^{-1}[(1 - \delta_{\mathbb{Z}}(k)) \cdot \Gamma_{\text{SSR}}^k]) \in L^q(G)$  for  $q \in (1, \frac{n}{n-1})$ . Recalling (3.5), we compute

$$[\delta_{ij}\mathfrak{R}_h \circ \mathfrak{R}_h - \mathfrak{R}_i \circ \mathfrak{R}_j](\mathcal{F}_{\mathbb{T}}^{-1}[(1 - \delta_{\mathbb{Z}}(k)) \cdot \Gamma_{\text{SSR}}^k]) = \Gamma_{ij}^\perp$$

and conclude (1.10).

In order to show (1.11), we further introduce

$$\Gamma_L : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}, \quad \Gamma_L := \begin{cases} -\frac{1}{2\pi} \log|x| & (n=2), \\ \frac{1}{(n-2)\omega_n} |x|^{2-n} & (n>2), \end{cases}$$

which is the fundamental solution to the Laplace equation  $\Delta \Gamma_L = \delta_{\mathbb{R}^n}$  in  $\mathbb{R}^n$ . As one may verify directly from the pointwise definitions of  $\Gamma_{\text{SSR}}^k$  and  $\Gamma_L$ , the convolution integral

$$\int_{\mathbb{R}^n} \Gamma_L(x-y) \Gamma_{\text{SSR}}^k(y) dy =: \Gamma_L * \Gamma_{\text{SSR}}^k(x) \quad (3.6)$$

exists for all  $x \in \mathbb{R}^n \setminus \{0\}$ . In fact, the function given by  $\Gamma_L * \Gamma_{SSR}^k$  belongs to  $L_{loc}^1(\mathbb{R}^n)$  and defines a tempered distribution in  $\mathcal{S}'(\mathbb{R}^n)$ . One may further verify that also the second order derivatives of  $\Gamma_L * \Gamma_{SSR}^k$  are given by convolution integrals

$$\partial_i \partial_j [\Gamma_L * \Gamma_{SSR}^k](x) = \int_{\mathbb{R}^n} \partial_i \Gamma_L(x-y) \partial_j \Gamma_{SSR}^k(y) dy, \quad (3.7)$$

from which it follows that their Fourier transform are functions

$$\mathcal{F}_{\mathbb{R}^n} [\partial_i \partial_j [\Gamma_L * \Gamma_{SSR}^k]](\xi) = \frac{\xi_i \xi_j}{|\xi|^2} \frac{1}{|\xi|^2 + i \frac{2\pi}{\mathcal{T}} k}.$$

We infer from the expression above that

$$\Gamma_{ij}^\perp = \mathcal{F}_{\mathbb{T}}^{-1} [(1 - \delta_{\mathbb{Z}}(k)) \cdot [\delta_{ij} \partial_h \partial_h - \partial_i \partial_j] [\Gamma_L * \Gamma_{SSR}^k]].$$

Employing Hausdorff-Young's inequality in combination with the pointwise estimate from Lemma 3.2 below, we obtain for  $r \in [2, \infty)$

$$\begin{aligned} \|\Gamma^\perp(x, \cdot)\|_{L^r(\mathbb{T})} &\leq \left( \sum_{k \in \mathbb{Z}} \left| (1 - \delta_{\mathbb{Z}}(k)) \cdot [\delta_{ij} \partial_h \partial_h - \partial_i \partial_j] [\Gamma_L * \Gamma_{SSR}^k](x) \right|^{r^*} \right)^{\frac{1}{r^*}} \\ &\leq C |x|^{-n} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-r^*} \right)^{\frac{1}{r^*}} \leq C |x|^{-n}, \end{aligned}$$

which concludes (1.11).

The convolution  $\Gamma^\perp * f$  can be expressed in terms of a Fourier multiplier

$$\Gamma^\perp * f = \mathcal{F}_G^{-1} \left[ M(\xi, k) \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \mathcal{F}_G[f] \right],$$

with  $M$  given by (3.2). As already mentioned,  $M \in L^\infty(\widehat{G})$ . As one may verify, also second order spatial derivatives  $\partial_i \partial_j M \in L^\infty(\widehat{G})$  and the time derivative  $\partial_t M \in L^\infty(\widehat{G})$  are bounded. Based on this information, (1.12) can be established. For the details of the argument, we refer the reader to [4, Proof of Theorem 4.8].  $\square$

**Lemma 3.1.** *The function  $\Gamma_{SSR}^k$  defined in (3.3) satisfies*

$$\left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |\Gamma_{SSR}^k(x)|^2 \right)^{\frac{1}{2}} \leq C |x|^{1-n} e^{-\frac{1}{2} \sqrt{\frac{\pi}{\mathcal{T}}} |x|}. \quad (3.8)$$

*Proof.* The estimates are based on the asymptotic properties of Hankel functions summarized in Lemma 3.3 below. We start with the case  $n > 2$ . Employing (3.16) with  $\varepsilon = 1$ , we deduce

$$\forall k \in \mathbb{Z} \quad \forall |x| \geq \sqrt{\frac{\mathcal{T}}{2\pi}} : \quad \left| H_{\frac{n}{2}-1}^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}} k} \cdot |x| \right) \right| \leq C |k|^{-\frac{1}{4}} |x|^{-\frac{1}{2}} e^{-\sqrt{\frac{\pi}{\mathcal{T}}} |k|^{\frac{1}{2}} |x|}. \quad (3.9)$$



Employing (3.17) with  $R = 1$ , we obtain:

$$\forall k \in \mathbb{Z} \ \forall |x| \leq \sqrt{\frac{\mathcal{T}}{2\pi}} |k|^{-\frac{1}{2}} : \quad \left| H_{\frac{n}{2}-1}^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}}} k \cdot |x| \right) \right| \leq C |k|^{-\frac{n-2}{4}} |x|^{-\frac{n-2}{2}}. \quad (3.10)$$

It follows that

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus \{0\}} |\Gamma_{\text{SSR}}^k(x)|^2 &\leq C \left( \sum_{|k| \leq \frac{\mathcal{T}}{2\pi} |x|^{-2}} |k|^{\frac{n-2}{2}} |x|^{2-n} |k|^{\frac{2-n}{2}} |x|^{2-n} \right. \\ &\quad \left. + \sum_{|k| > \frac{\mathcal{T}}{2\pi} |x|^{-2}} |k|^{\frac{n-2}{2}} |x|^{2-n} |k|^{-\frac{1}{2}} |x|^{-1} e^{-2\sqrt{\frac{\pi}{\mathcal{T}}} |k|^{\frac{1}{2}} |x|} \right) \\ &\leq C \left( |x|^{-2} \cdot |x|^{2(2-n)} \cdot \chi_{[0, \sqrt{\frac{\mathcal{T}}{2\pi}}]}(|x|) \right. \\ &\quad \left. + \sum_{|k| \geq 1} |k|^{\frac{n-3}{2}} |x|^{1-n} e^{-2\sqrt{\frac{\pi}{\mathcal{T}}} |k|^{\frac{1}{2}} |x|} \right). \end{aligned} \quad (3.11)$$

For  $|q| < 1$  we observe that

$$\begin{aligned} \sum_{k \geq 1} |k|^{\frac{n-3}{2}} q^{k^{\frac{1}{2}}} &= \sum_{j=1}^{\infty} \sum_{k=j^2}^{(j+1)^2-1} k^{\frac{n-3}{2}} q^{k^{\frac{1}{2}}} \\ &\leq \sum_{j=1}^{\infty} \sum_{k=j^2}^{(j+1)^2-1} (j+1)^{n-3} q^j \\ &= \sum_{j=1}^{\infty} j (j+1)^{n-3} q^j \\ &\leq q \sum_{j=1}^{\infty} j (j+1) (j+2) \dots (j+n-3) q^{j-1} \\ &= q \partial_q^{n-2} \left[ \sum_{j=1}^{\infty} q^{j+n-3} \right] = q \partial_q^{n-2} [(1-q)^{-1}] \\ &= (n-2)! \cdot q (1-q)^{1-n}, \end{aligned}$$

from which we deduce

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus \{0\}} |\Gamma_{\text{SSR}}^k(x)|^2 &\leq C \left( |x|^{2(1-n)} \cdot \chi_{[0, \sqrt{\frac{\mathcal{T}}{2\pi}}]}(|x|) \right. \\ &\quad \left. + |x|^{1-n} e^{-2\sqrt{\frac{\pi}{\mathcal{T}}} |x|} (1 - e^{-2\sqrt{\frac{\pi}{\mathcal{T}}} |x|})^{1-n} \right) \\ &\leq C |x|^{2(1-n)} \cdot e^{-\sqrt{\frac{\pi}{\mathcal{T}}} |x|} \end{aligned}$$

and consequently (3.8) in the case  $n > 2$ . In the case  $n = 2$ , we employ (3.18) to deduce

$$\forall k \in \mathbb{Z} \ \forall |x| \leq \sqrt{\frac{\mathcal{T}}{2\pi}} |k|^{-\frac{1}{2}} : \quad \left| H_0^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}}} k \cdot |x| \right) \right| \leq C \left| \log \left( \sqrt{\frac{2\pi}{\mathcal{T}}} |k|^{\frac{1}{2}} |x| \right) \right|. \quad (3.12)$$

It follows in the case  $n = 2$  that<sup>2</sup>

$$\begin{aligned} \sum_{|k| \leq \frac{\mathcal{T}}{2\pi} |x|^{-2}} |\Gamma_{\text{SSR}}^k(x)|^2 &\leq C \sum_{|k| \leq \frac{\mathcal{T}}{2\pi} |x|^{-2}} \left| \log \left( \sqrt{\frac{2\pi}{\mathcal{T}}} |k|^{\frac{1}{2}} |x| \right) \right|^2 \\ &\leq C \int_0^{\frac{\mathcal{T}}{2\pi} |x|^{-2}} \left| \log \left( \sqrt{\frac{2\pi}{\mathcal{T}}} t^{\frac{1}{2}} |x| \right) \right|^2 dt \cdot \chi_{[0, \sqrt{\frac{\mathcal{T}}{2\pi}}]}(|x|) \\ &\leq C |x|^{-2} \int_0^1 |\log(s)|^2 s ds \cdot \chi_{[0, \sqrt{\frac{\mathcal{T}}{2\pi}}]}(|x|) \\ &\leq C |x|^{-2} \cdot \chi_{[0, \sqrt{\frac{\mathcal{T}}{2\pi}}]}(|x|). \end{aligned} \quad (3.13)$$

Estimate (3.9) is still valid in the case  $n = 2$ . We can thus proceed as in (3.11) and obtain (3.8) also in the case  $n = 2$ .  $\square$

**Lemma 3.2.** *The convolution  $\Gamma_L * \Gamma_{\text{SSR}}^k$  defined in (3.6) satisfies*

$$\forall \varepsilon > 0 \ \exists C > 0 \ \forall |x| \geq \varepsilon : \quad |\partial_i \partial_j [\Gamma_L * \Gamma_{\text{SSR}}^k](x)| \leq C |k|^{-1} |x|^{-n}. \quad (3.14)$$

*Proof.* Fix  $\varepsilon > 0$  and consider some  $x \in \mathbb{R}^n$  with  $|x| \geq \varepsilon$ . Put  $R := \frac{|x|}{2}$ . Let  $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R})$  be a “cut-off” function with

$$\chi(r) = \begin{cases} 0 & \text{when } 0 \leq |r| \leq \frac{1}{2}, \\ 1 & \text{when } 1 \leq |r| \leq 3, \\ 0 & \text{when } 4 \leq |r|. \end{cases}$$

Define  $\chi_R : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\chi_R(y) := \chi(R^{-1}|y|)$ . We use  $\chi_R$  to decompose the integral in (3.7) as

$$\begin{aligned} \partial_i \partial_j [\Gamma_L * \Gamma_{\text{SSR}}^k](x) &= \int_{B_{4R, R/2}} \partial_i \Gamma_L(x-y) \partial_j \Gamma_{\text{SSR}}^k(y) \chi_R(y) dy \\ &\quad + \int_{\bar{B}_R} \partial_i \Gamma_L(x-y) \partial_j \Gamma_{\text{SSR}}^k(y) (1 - \chi_R(y)) dy \\ &\quad + \int_{B^{3R}} \partial_i \Gamma_L(x-y) \partial_j \Gamma_{\text{SSR}}^k(y) (1 - \chi_R(y)) dy \\ &=: I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

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<sup>2</sup>I would like to thank Prof. Toshiaki Hishida for suggesting this estimate to me and thereby improving my original proof.

Recalling the definition (3.3) of  $\Gamma_{\text{SSR}}^k$  as well as the property (3.15) and the estimate (3.16) of the Hankel function, we can estimate for  $|y| \geq R/2$ :

$$\begin{aligned} |\partial_j \Gamma_{\text{SSR}}^k(y)| &\leq C |k|^{\frac{n-2}{4}} \left( \left| \partial_j \left[ |y|^{\frac{2-n}{2}} \right] H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}}} k \cdot |y| \right) \right| \right. \\ &\quad \left. + \left| |y|^{\frac{2-n}{2}} \partial_j \left[ H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}}} k \cdot |y| \right) \right] \right| \right) \\ &\leq C \left( |k|^{\frac{n}{4}-\frac{3}{4}} |y|^{-\frac{n}{2}-\frac{1}{2}} + |k|^{\frac{n}{4}-\frac{1}{4}} |y|^{-\frac{n}{2}+\frac{1}{2}} \right) e^{-\sqrt{\frac{\pi}{\mathcal{T}}} |k|^{\frac{1}{2}} |y|} \\ &\leq C |k|^{-1} |y|^{-(n+1)}. \end{aligned}$$

Consequently, we obtain:

$$|I_1(x)| \leq C \int_{B_{4R, R/2}} |x-y|^{1-n} |k|^{-1} |y|^{-(n+1)} dy \leq C |k|^{-1} R^{-n}.$$

To estimate  $I_2$ , we integrate partially and employ polar coordinates to deduce

$$\begin{aligned} |I_2(x)| &\leq C \int_{B_R} |\partial_j \partial_i \Gamma_L(x-y)| |\Gamma_{\text{SSR}}^k(y)| + |\partial_i \Gamma_L(x-y)| |\Gamma_{\text{SSR}}^k(y)| R^{-1} dy \\ &\leq C \int_{B_R} R^{-n} |\Gamma_{\text{SSR}}^k(y)| dy \\ &\leq C \int_{B_R} R^{-n} |k|^{\frac{n-2}{4}} |y|^{\frac{2-n}{2}} \left| H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}}} k \cdot |y| \right) \right| dy \\ &\leq C \int_0^R R^{-n} |k|^{\frac{n-2}{4}} r^{\frac{n}{2}} \left| H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}}} k \cdot r \right) \right| dr \\ &\leq C \int_0^\infty R^{-n} |k|^{-1} s^{\frac{n}{2}} \left| H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{\mathcal{T}}} \cdot \sqrt{\frac{k}{|k|}} \cdot s \right) \right| ds. \end{aligned}$$

Employing in the case  $n > 2$  estimate (3.17) in combination with (3.16), we obtain

$$|I_2(x)| \leq C R^{-n} |k|^{-1} \left( \int_0^1 s^{\frac{n}{2}} s^{\frac{2-n}{2}} ds + \int_1^\infty s^{\frac{n}{2}} s^{-\frac{1}{2}} e^{-\sqrt{\frac{\pi}{\mathcal{T}}} s} ds \right) \leq C R^{-n} |k|^{-1}.$$

When  $n = 2$ , we use estimate (3.18) in combination with (3.16) and obtain also in this case

$$|I_2(x)| \leq C R^{-n} |k|^{-1} \left( \int_0^1 s \cdot \left| \log \left( \sqrt{\frac{\pi}{\mathcal{T}}} s \right) \right| ds + \int_1^\infty s^{\frac{1}{2}} e^{-\sqrt{\frac{\pi}{\mathcal{T}}} s} ds \right) \leq C R^{-n} |k|^{-1}.$$

In order to estimate  $I_3$ , we again integrate partially and utilize (3.16):

$$\begin{aligned}
|I_3(x)| &\leq C \int_{B^{3R}} |\partial_j \partial_i \Gamma_L(x-y)| |\Gamma_{\text{SSR}}^k(y)| + |\partial_i \Gamma_L(x-y)| |\Gamma_{\text{SSR}}^k(y)| R^{-1} dy \\
&\leq C \int_{B^{3R}} R^{-n} |\Gamma_{\text{SSR}}^k(y)| dy \\
&\leq C \int_{B^{3R}} R^{-n} |k|^{\frac{n-2}{4}} |y|^{\frac{2-n}{2}} \left| H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{T}} k \cdot |y| \right) \right| dy \\
&\leq C \int_{B^{3R}} R^{-n} |k|^{\frac{n-3}{4}} |y|^{\frac{1-n}{2}} e^{-\sqrt{\frac{\pi}{T}} |k|^{\frac{1}{2}} |y|} dy \\
&\leq C \int_{B^{3R}} R^{-n} |k|^{\frac{n-3}{4}} |y|^{\frac{1-n}{2}} (|k|^{\frac{1}{2}} |y|)^{-\frac{n+3}{2}} dy \leq C R^{-n} |k|^{-\frac{3}{2}} \leq C R^{-n} |k|^{-1}.
\end{aligned}$$

Since  $|x| = 2R$ , we conclude (3.14) by collecting the estimates for  $I_1$ ,  $I_2$  and  $I_3$ .  $\square$

**Lemma 3.3.** *Hankel functions are analytic in  $\mathbb{C} \setminus \{0\}$  with*

$$\forall \nu \in \mathbb{C} \quad \forall z \in \mathbb{C} \setminus \{0\} : \quad \frac{d}{dz} H_\nu^{(1)}(z) = H_{\nu-1}^{(1)}(z) - \frac{\nu}{z} H_\nu^{(1)}(z). \quad (3.15)$$

*The Hankel functions satisfy the following estimates:*

$$\forall \nu \in \mathbb{C} \quad \forall \varepsilon > 0 \quad \exists C > 0 \quad \forall |z| \geq \varepsilon : \quad |H_\nu^{(1)}(z)| \leq C |z|^{-\frac{1}{2}} e^{-\text{Im } z}, \quad (3.16)$$

$$\forall \nu \in \mathbb{R}_+ \quad \forall R > 0 \quad \exists C > 0 \quad \forall |z| \leq R : \quad |H_\nu^{(1)}(z)| \leq C |z|^{-\nu}, \quad (3.17)$$

$$\forall R > 0 \quad \exists C > 0 \quad \forall |z| \leq R : \quad |H_0^{(1)}(z)| \leq C |\log(|z|)|. \quad (3.18)$$

*Proof.* The recurrence relation (3.15) is a well-know property of various Bessel functions; see for example [1, 9.1.27]. We refer to [1, 9.2.3] for the asymptotic behaviour (3.16) of  $H_\nu^{(1)}(z)$  as  $z \rightarrow \infty$ . See [1, 9.1.9 and 9.1.8] for the asymptotic behaviour (3.17) and (3.18) of  $H_\nu^{(1)}(z)$  as  $z \rightarrow 0$ .  $\square$

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